Correlation Functions for Random Involutions

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Abstract

Our interest is in the scaled joint distribution associated with k-increasing subsequences for random involutions with a prescribed number of fixed points. We proceed by specifying in terms of correlation functions the same distribution for a Poissonized model in which both the number of symbols in the involution, and the number of fixed points, are random variables. From this, a de-Poissonization argument yields the scaled correlations and distribution function for the random involutions. These are found to coincide with the same quantities known in random matrix theory from the study of ensembles interpolating between the orthogonal and symplectic universality classes at the soft edge, the interpolation being due to a rank 1 perturbation.

1 Introduction

To motivate our study of random involutions, we first recall the corresponding problem for random permutations, the solution of which is known. Let S_N denote the set of the $N!$ distinct permutations of $\{1,2,\dots,N\}$. For each $\pi \in S_N$ denote the image of the number $i \in \{1,2,\dots,N\}$ by $\pi(i)$. A subsequence of image points $\pi(i_1),\pi(i_2),\dots,\pi(i_j)$ where $1 \le i_1 < \cdots < i_j \le N$ is said to be an increasing subsequence of length j if $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_j)$. More generally, we say there is a k-increasing subsequence of length j if π contains k disjoint subsequences

$$\pi(i_1^{(l)}) < \pi(i_2^{(l)}) < \cdots < \pi(i_{j_l}^{(l)}), \quad 1 \le i_1^{(l)} < \cdots < i_{j_l}^{(l)} \le N \ \ (l=1,\dots,k)$$

with $\sum_{l=1}^k j_l = j$. Being disjoint these subsequences contain no common member. For a given π, let $L_N^{(k)}(\pi)$ denote the maximum length of all the k-increasing subsequences and define

$$\lambda_N^{(k)}(\pi) = L_N^{(k)}(\pi) - L_N^{(k-1)}(\pi), \qquad L_N^{(-1)}(\pi) := 0. \tag{1.1}$$

Note that

$$\lambda_N^{(1)}(\pi) \ge \lambda_N^{(2)}(\pi) \ge \cdots \ge \lambda_N^{(N)}(\pi) \ge 0.$$

Consider an ensemble of permutations of $\{1,2,\dots,N\}$ in which each permutation is equally likely. The problem of interest is the computation of the scaled joint distribution of $\{\lambda_N^{(j)}(\pi)\}_{j=1,\dots,l}$ in the limit $N \to \infty$.

In the case $l=1$ it was proved by Baik, Deift and Johansson [4] that

$$\lim_{N\to\infty} \Pr\Big(\frac{\lambda_N^{(1)} - 2\sqrt{N}}{N^{1/6}} \le s\Big) = F_2(s), \tag{1.2}$$

where $F_2(s)$ is the scaled cumulative distribution of the largest eigenvalue for large random Hermitian matrices with complex elements (technically matrices from the Gaussian unitary ensemble (GUE)) [10, 29]. The latter is specified in terms of a Fredholm determinant according to

$$F_2(s) = \det(1 - K^{\rm soft}\chi_{(s,\infty)}),$$

where χ_J is the characteristic function of the interval J, while $K^{\rm soft}$ is the integral operator with kernel given in terms of Airy functions according to

$$\begin{aligned} K^{\rm soft}(x,y) &= \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y} \\ &= \int_0^\infty \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)\,dt. \end{aligned} \tag{1.3}$$

(The superscript 'soft' indicates that the eigenvalue density $\rho_{(1)}(s)$ is not strictly zero for any s, even though the origin is chosen in the neighbourhood of the largest eigenvalue and thus at the spectrum edge). The case of general l was solved in [7, 19, 24], where it was proved that

$$\lim_{N\to\infty} \Pr\Big(\frac{\lambda_N^{(1)} - 2\sqrt{N}}{N^{1/6}} \le s_1, \dots, \frac{\lambda_N^{(l)} - 2\sqrt{N}}{N^{1/6}} \le s_l\Big) = F_2(s_1,\dots,s_l).$$

Here $F_2(s_1,\dots,s_l)$ is the scaled joint distribution of the l largest eigenvalues in the GUE. The latter is uniquely specified in terms of the k-point correlation function

$$\rho_{(k)}^{\rm scaled}(s_1,\dots,s_k) = \det[K^{\rm soft}(s_j,s_l)]_{j,l=1,\dots,k}$$

for the scaled eigenvalues at the soft edge of the GUE.

It is the objective of this study to calculate the analogous joint probability for involutions with a prescribed number of fixed points. One recalls that an involution is a permutation π with the additional property that $\pi^2 = I$, where I denotes the identity permutation. Involutions must consist entirely of two cycles and fixed points. Thus for an involution of $\{1, 2, \dots, N\}$, if there are n two cycles, there must be $m = N - 2n$ fixed points. As emphasized in [2], a random involution with a prescribed number of fixed points can be generated geometrically by marking n points ($n \le [N/2]$) in the unit square below the diagonal $y = x$ uniformly at random, marking the images of these points under reflection about $y = x$, and marking $N - 2n$ points uniformly at random on the diagonal. Equivalently the unit square can first be divided into a $N \times N$ integer grid, and the points marked at random on the lattice sites below and on the diagonal according to the above prescription, with the additional constraint that no two points are in the same row or column. Either way, projecting the points onto the x-axis gives a sequence of x co-ordinates $x_1 < x_2 < \cdots < x_N$ while projecting them onto the y-axis gives a sequence of y co-ordinates $y_1 < y_2 < \cdots < y_N$. Each point will then have a co-ordinate $(x_i, y_{\pi(i)})$ with the property that $\pi(i) = i$ for the point on the diagonal, and that $(x_{\pi(i)}, y_i)$ is the point reflected in the diagonal otherwise. Hence π defines an involution with a prescribed number of fixed points, and furthermore the involutions are generated at random with uniform probability by this procedure.

The quantity $L_N^{(k)}(\pi)$ admits an interpretation in the above setting. Connect points by segments which always have positive slope to form a continuous path, which is said to be right/diagonal (rd). Define the length of this path as the number of points it contains, and denote it by #rd. Similarly, let $(\mathrm{RD})^k$ denote the set of all k disjoint rd lattice paths, formed from amongst the points with a path being disjoint if it contains no common points, and for $(\mathrm{rd})^k \in (\mathrm{RD})^k$ let $\#(\mathrm{rd})^k$ denote the number of lattice points. Then by considering recurrences satisfied by the various quantities one can show [26]

$$L_N^{(k)}(\pi) = \max \sum_{(\mathrm{rd})^k \in (\mathrm{RD})^k} \#(\mathrm{rd})^k \tag{1.4}$$

(for the recurrences satisfied by the left hand side, see [12, Appendix A]). The equation (1.4) holds for points corresponding to a general permutation π. In the case that π is an involution, it is easy to see that the set of lattice paths $(\mathrm{RD})^k$ can be restricted to those which contain points on or below the diagonal (see Figure 1 for an example).

The explicit form of the scaled distribution of $L_N^{(1)} =: \lambda_{n,m}^{(1)}$ for random involutions with n 2-cycles and m fixed points, and thus the solution to our problem in the case $l = 1$, is already known [3]. Thus introduce the scaling variable w by the requirement that

$$m = [\sqrt{2n} - 2w(2n)^{1/3}], \tag{1.5}$$

where $[\cdot]$ denotes the integer part. It is proved in [3] that with w fixed

$$\lim_{N\to\infty} \Pr\Big(\frac{\lambda_{n,m}^{(1)} - 2\sqrt{N}}{N^{1/6}} \le s \Big) = F^{\boxslash}(s; w), \tag{1.6}$$

where the distribution $F^{\boxslash}(s; w)$ is specified in terms of a certain Riemann-Hilbert problem related to the Painlevé II equation with special monodromy data. This distribution has the property that

$$F^{\boxslash}(s; 0) = F_1(s), \qquad \lim_{w\to\infty} F^{\boxslash}(s; w) = F_4(s),$$

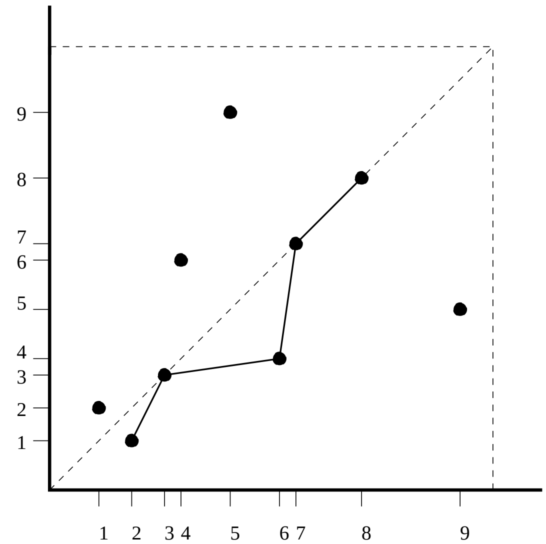

Figure 1: Nine points in the square symmetric about the diagonal which correspond to the involution (12)(3)(46)(59)(7)(8). An rd path realizing $L_N^{(1)}(\pi)$ has been drawn.

where $F_1(s), F_4(s)$ are the scaled cumulative distributions for the largest eigenvalue in the GOE of random real symmetic matrices, and the GSE of Hermitian matrices with real quaternion elements respectively.

Here we seek the joint distribution of $\{\lambda_{n,m}^{(j)}(\pi)\}_{j=1,\dots,l}$, scaled as in (1.5), (1.6) for general l. Regarding the joint distribution of all the $\lambda_{n,m}^{(j)}(\pi)$, $j = 1, \dots, N$ as specifying a point process, it is generally true that the distribution of the l right-most points is fully determined by the corresponding correlation functions. The specification of the correlations for a point process implying the distribution $F^{\boxslash}(s; w)$ for the right-most point has recently been given [11]. This was found in the study of a closely related geometrical model to that corresponding to random involutions and their increasing subsequences. Thus consider an $M \times M$ integer grid. Associate with each lattice site on or below the diagonal a continuous exponential variable

$$\begin{aligned}
\Pr(x_{i,j} \in [y, y + dy]) &= e^{-y}\, dy, \qquad i < j, \\
\Pr(x_{i,i} \in [y, y + dy]) &= \frac{(1 - A)}{2} e^{-(1-A)y/2}\, dy,
\end{aligned} \tag{1.7}$$

and impose the symmetry constraint that $x_{i,j} = x_{j,i}$ for $i > j$. The quantities (1.4), with the lattice points counted according to their weightings $x_{i,j}$, are well defined. Setting then $x_k = L_M^{(k)} - L_M^{(k-1)}$, $L_M^{(-1)} := 0$ we have $x_1 > x_2 > \cdots > x_M > 0$ and furthermore the joint distribution of these variables is proportional to [1, Section 3] [12, Prop. 4]

$$e^{-\sum_{j=1}^M x_j/2} e^{A \sum_{j=1}^M (-1)^{j-1} x_j/2} \prod_{1 \le i < j \le M} (x_i - x_j). \tag{1.8}$$

This same p.d.f. occurs in random matrix theory. Thus let X be a $2n \times 2n$ antisymmetric complex Gaussian matrix (independent entries distributed as $\mathrm{N}[0, 1] + i\mathrm{N}[0, 1]$), and let $\vec{x}$ be a $2n \times 1$ complex Gaussian vector with entries distributed as $\mathrm{N}[0, 1/\sqrt{2}] + i\mathrm{N}[0, 1/\sqrt{2}]$. Then we know from [12, Thm. 3] that $Y = X^\dagger X + b\vec{x}\vec{x}^\dagger$ has eigenvalue p.d.f. (1.8) with $A = 1 - 2/b$, $M = 2n$.

For the p.d.f. (1.8), the scaled correlation functions with

$$x_j = 4M + 2(2M)^{1/3} X_j, \qquad A = u/2(2M)^{1/3}, \tag{1.9}$$

and $M \to \infty$ have been computed [11] (actually the symbol α is used instead of u in [11], but we use α for another purpose below). On the other hand, a result of Baik [1] gives

$$\lim_{M\to\infty} \Pr\Big(\frac{L_M^{(1)}|_{A=-2w/(2M)^{1/3}} - 4M}{2(2M)^{1/3}} \le s \Big) = F^{\boxslash}(s;w). \tag{1.10}$$

The scalings (1.9) is precisely that in (1.10) with the identification

$$w = -\frac{u}{4}, \tag{1.11}$$

so this distribution is fully determined by the correlations computed in [11]. We therefore expect that for random involutions, the generalization of (1.6) is

$$\lim_{N\to\infty} \Pr\Big(\frac{\lambda_{n,m}^{(1)} - 2\sqrt{N}}{N^{1/6}} \le s_1, \dots, \frac{\lambda_{n,m}^{(l)} - 2\sqrt{N}}{N^{1/6}} \le s_l \Big) = F^{\boxslash}(s_1,\dots,s_l;w), \tag{1.12}$$

where $F^{\boxslash}(s_1,\dots,s_l;w)$ is the joint distribution of the scaled l right-most points in the process specified by (1.8), scaled as in (1.9) and with u and w related by (1.11). As revised below, it is generally true that the latter distribution is fully determined by the scaled correlation functions $\rho_{(k)}^{\rm scaled}$. According to [11] these have the explicit form

$$\rho_{(k)}^{\rm scaled}(X_1,\dots,X_k;u) = \mathrm{qdet}\,[f(X_i,X_j)]_{i,j=1,\dots,k}, \tag{1.13}$$

where with $\mathrm{sgn}(u) = 1,\ -1,\ 0$ for $u>0$, $u<0$, $u=0$ respectively, f is a 2×2 matrix with entries

$$\begin{aligned}
f^{11}(X,Y) &= f^{22}(Y,X), \\
f^{22}(X,Y) &= \frac{1}{2}K^{\rm soft}(X,Y) - \frac{1}{2}\frac{\partial}{\partial Y}\int_{-\infty}^{X} e^{u(X-t)/2} K^{\rm soft}(t,Y)\,dt \\
&\quad -\frac{u}{4}\int_{-\infty}^{X} dt\, e^{u(X-t)/2}\int_{Y}^{\infty} ds\,\frac{\partial}{\partial t}K^{\rm soft}(s,t), \\
f^{12}(X,Y) &= \frac{1}{4}\Big(\frac{u}{2}+\frac{\partial}{\partial X}\Big)\Big(\frac{u}{2}+\frac{\partial}{\partial Y}\Big)\Big\{\int_{X}^{\infty} K^{\rm soft}(Y,t)\,dt - \int_{Y}^{\infty} K^{\rm soft}(X,t)\,dt\Big\}, \\
f^{21}(X,Y) &= -e^{u|X-Y|/2}\mathrm{sgn}(X-Y) - \Big\{\int_{-\infty}^{Y} e^{u(Y-t)/2}K^{\rm soft}(X,t)\,dt \\
&\quad -\int_{-\infty}^{X} e^{u(X-t)/2}K^{\rm soft}(Y,t)\,dt\Big\},
\end{aligned} \tag{1.14}$$

and the qdet operation is defined in terms of the more familiar Pfaffian by the general formula

$$\mathrm{qdet}\,A = \mathrm{Pf}(AZ_{2n}^{-1}), \qquad Z_{2n} = 1_n \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

valid for all $2n\times 2n$ matrices A with the self dual property $A = Z_{2n}^{-1}A^T Z_{2n}$. Our main task then is to show that the same scaled correlations determine the joint distribution of $\{\lambda_{n,m}^{(j)}(\pi)\}_{j=1,\dots,l}$. We will see that doing this indeed allows (1.12) to be validated, giving us the following limit theorem, which is our main result.

Theorem 1. *Consider an involution on $N = 2n + m$ symbols, consisting of m fixed points and n 2-cycles, chosen at random with uniform distribution on the set of such involutions. Let $\lambda^{(k)}_{n,m}$ be specified in terms of the maximum length of the k-increasing subsequences $L^{(k)}_N$ according to (1.1). With m related to n by (1.5), the limit formula (1.12) holds.*

2 Random Generalized Involutions

2.1 Strategy

We are guided in our approach by previous studies on the calculation of distribution functions relating to increasing subsequences for random permutations [4, 7, 18], and also previous studies on the calculation of the distribution of the maximum increasing subsequence for random involutions [3, 6]. For both random permutations and random involutions, all the studies proceed by Poissonizing the ensemble. In the case of random permutations, the number of symbols N is itself taken as a variable which occurs with probability $e^{-z}z^N/N!$. For random involutions the number of two cycles n and number of fixed points m are separately taken as random variables, occurring with probability $e^{-z_1-z_2}z_1^n z_2^m/n!m!$. One then seeks to calculate the distribution function of interest in the Poissonized ensemble, and to compute the $z \to \infty$ $(z_1, z_2 \to \infty)$ scaled limit. A de-Poissonization argument [17] gives that the latter is equivalent to the $N \to \infty$ $(n, m \to \infty)$ limit in the original ensemble.

As the distribution functions are controlled by the correlation functions $\rho_{(k)}$, one would like to first calculate the Poissonized correlation functions. However we know from studies of random permutations that a direct calculation is not practical. Instead, as was shown by Johansson [18], progress can be made by constructing the Poissonization as a limiting case of a generalization of the geometrical viewpoint of the increasing subsequence problem. In one such generalization, giving rise to the so called Meixner ensemble [19], each lattice point of an $M \times M$ grid carries a non-negative integer variable chosen from the geometric distribution with parameter q. For this model $\rho_{(k)}$ can be computed in terms of certain orthogonal polynomials. Taking the limit $M \to \infty$ with $q = Q/M^2$ gives $\rho_{(k)}$ for the Poissonized version of the original model. Thus we must first calculate $\rho_{(k)}$ for the version of the Meixner ensemble relevant to involutions.

2.2 The joint p.d.f., correlations and distribution functions

According the above strategy, our first task is to introduce a generalization of the geometrical model corresponding to random involutions and their increasing subsequences. For this we consider an $M \times M$ grid, and associate with each lattice site on or below the diagonal a non-negative integer variable

$$\begin{aligned} \Pr(x_{i,j} = k) &= (1-q)q^k, \qquad i < j, \\ \Pr(x_{i,i} = k) &= (1-\sqrt{\alpha q})(\alpha q)^{k/2}. \end{aligned} \tag{2.1}$$

For $i > j$ we impose the symmetry constraint $x_{i,j} = x_{j,i}$. Notice that the symmetry constraints of this model are the same as that for the geometrical model of random involutions specified in the Introduction, and notice too the similarity with the model defined by (1.7) and surrounding text. As with the latter model, the quantities (1.4), which can be regarded as a sequence of last passage times, are well defined. With $\lambda_k := L^{(k)}_M - L^{(k-1)}_M$, $L^{(-1)}_M := 0$, we know from [1, Section

3], [12, Prop. 1] that the joint distribution of the latter variables have the explicit form

$$P_M(\lambda;q) = (1-\sqrt{\alpha q})^M (1-q)^{M(M-1)/2} q^{\sum_{j=1}^M \lambda_j/2} \alpha^{\sum_{j=1}^M (-1)^{j-1}\lambda_j/2} \prod_{1\le j<l\le M} \frac{\lambda_j-\lambda_l+l-j}{l-j}. \tag{2.2}$$

In terms of $h_j := \lambda_j + M - j$ this reads

$$P(h_1,\cdots,h_M) = C_M(q,\alpha) q^{\sum_{j=1}^M h_j/2} \alpha^{\sum_{j=1}^M (-1)^{j-1} h_j} \prod_{1\le j<l\le M} (h_j - h_l), \tag{2.3}$$

where

$$C_M(q,\alpha) = (1-\sqrt{\alpha q})^M (1-q)^{M(M-1)/2} q^{-M(M-1)/2} \alpha^{-M/4} \prod_{j=1}^{M-1} (1/j!) \tag{2.4}$$

and $\infty > h_1 > h_2 > \cdots > h_M \ge 0$. Note the similarity between (2.3) and (1.8).

Let us introduce the symmetrized joint p.d.f. by

$$P_{\rm sym}(h_1,\ldots,h_M) = \sum_{\mu\in S_M} P(h_{\mu(1)},\ldots,h_{\mu(M)}) \chi_{h_{\mu(1)}>\cdots>h_{\mu(N)}}, \tag{2.5}$$

where $\chi_T = 1$ for T true, and $\chi_T = 0$ otherwise. The k-point correlation $\rho_{(k)}$ is then given by

$$\rho_{(k)}(h_1,\ldots,h_k) = \frac{1}{(M-k)!} \sum_{h_{k+1},\ldots,h_M=0}^{\infty} P(h_1,\ldots,h_M). \tag{2.6}$$

The correlations (2.6) can be used to compute the distribution functions $\Pr(h_1 \le a_1,\ldots,h_l \le a_l)$ for the joint p.d.f. (2.3). To see this, let $a_0 = \infty$, let $a_1 > a_2 > \cdots > a_l$ be non-zero integers, and put $I_j = (a_j, a_{j-1})$ where (a_j, a_{j-1}) denotes all integers between (but not including) a_j and a_{j-1}. Let $E_M(\{(n_r, I_r)\}_{r=1,\ldots,l})$ denote the probability that n_r of the coordinates $\{h_i\}_{i=1,\ldots,M}$ are in I_r $(r = 1,\ldots,l)$. Then as a consequence of the definitions we know from [19, eq. (3.41)] that with

$$\mathbb{L}_l := \{(n_1,\ldots,n_l) \in \mathbb{Z}^l_{\ge 0} \,:\, \sum_{j=1}^r n_j \le r-1 \ (r=1,\ldots,l)\},$$

one has

$$\Pr(h_1 \le a_1,\ldots,h_l \le a_l) = \sum_{(n_1,\ldots,n_l)\in\mathbb{L}_l} E_M(\{(n_r,I_r)\}_{r=1,\ldots,l}). \tag{2.7}$$

Furthermore, again from the definitions, it is easy to see that

$$E_M(\{(n_r,I_r)\}_{r=1,\ldots,l}) = \frac{(-1)^{\sum_{r=1}^l n_r}}{n_1!\cdots n_l!} \frac{\partial^{\sum_{j=1}^l n_j}}{\partial\xi^{n_1}\cdots\partial\xi^{n_l}} \Big\langle \prod_{j=1}^M \Big(1 - \sum_{r=1}^k \xi_r \chi_{I_r}^{(j)}\Big) \Big\rangle_{\rm Psym}\Big|_{\xi_1=\cdots=\xi_k=1}, \tag{2.8}$$

where $\chi_{I_r}^{(j)} = 1$ if $h_j \in I_r$, $\chi_{I_r}^{(j)} = 0$ otherwise, and that the average herein is given in terms of the correlations (2.6) according to

$$\Big\langle \prod_{j=1}^M \Big(1 - \sum_{r=1}^k \xi_r \chi_{I_r}^{(j)}\Big) \Big\rangle_{\rm Psym} = 1 + \sum_{p=1}^M \frac{(-1)^p}{p!} \sum_{h_1,\ldots,h_p=0}^{\infty} \prod_{i=1}^p \Big(\sum_{r=1}^k \xi_r \chi_{I_r}^{(i)}\Big) \rho_{(p)}(h_1,\ldots,h_p). \tag{2.9}$$

2.3 Quaternion determinant expression for the k-point correlation

We know from [11, Eq. (3.1)] that with P given by (2.3), and M even (2.5) can be written

$$P_{\rm sym}(h_1,\dots,h_M) = C_M(q,\alpha) q^{\sum_{j=1}^M h_j/2} \prod_{j>l}^{M} (h_j - h_l) {\rm Pf}[\epsilon(h_j,h_l)]_{j,l=1,2,\cdots,M}, \tag{2.10}$$

where

$$\epsilon(x,y) = \alpha^{|y-x|/2} {\rm sgn}(y-x) \tag{2.11}$$

(a similar formula can be given in the case that M odd, but for definiteness we will proceed with the assumption that M is even). This structure is familiar in random matrix theory, and in the theory of random measures on partitions relating to increasing subsequences [27]. It is known [14, 22] that in general the corresponding k-point correlations have the quaternion determinant form

$$\rho_{(k)}(h_1,\cdots,h_k) = \ {\rm qdet}[f(h_j,h_l)]_{j,l=1,2,\cdots,k}, \tag{2.12}$$

where $f(x,y)$ is the 2×2 matrix

$$f(x,y) = \begin{bmatrix} S(x,y) & I(x,y) \\ D(x,y) & S(y,x) \end{bmatrix}. \tag{2.13}$$

According to the general formalism, to specify the matrix elements in (2.13) we must introduce skew orthogonal polynomials $R_n(x)$ satisfying

$$\langle R_{2m}(x), R_{2n+1}(y)\rangle = -\langle R_{2n+1}(x), R_{2m}(y)\rangle = r_m \delta_{m,n},$$

$$\langle R_{2m}(x), R_{2n}(y)\rangle = 0, \quad \langle R_{2m+1}(x), R_{2n+1}(y)\rangle = 0, \tag{2.14}$$

where

$$\begin{aligned} \langle f(x), g(y)\rangle &= \frac{1}{2}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} q^{(x+y)/2}\epsilon(x,y)(f(x)g(y)-f(y)g(x)) \\ &= \sum_{y=0}^{\infty} q^{y/2}\alpha^{-y/2} f(y) \sum_{x=y+1}^{\infty} q^{x/2}\alpha^{x/2} g(x) - \sum_{y=0}^{\infty} q^{y/2}\alpha^{-y/2} g(y) \sum_{x=y+1}^{\infty} q^{x/2}\alpha^{x/2} f(x). \end{aligned} \tag{2.15}$$

One then has

$$f(x,y) = \begin{bmatrix} S(x,y) & I(x,y) \\ D(x,y) & S(y,x) \end{bmatrix}, \tag{2.16}$$

where

$$S(x,y) = q^{y/2} \sum_{j=0}^{(M/2)-1} \frac{1}{r_j}\left[\Phi_{2j}(x) R_{2j+1}(y) - \Phi_{2j+1}(x) R_{2j}(y)\right], \tag{2.17}$$

$$I(x,y) = -\sum_{j=0}^{(M/2)-1} \frac{1}{r_j}\left[\Phi_{2j}(x)\Phi_{2j+1}(y) - \Phi_{2j+1}(x)\Phi_{2j}(y)\right] + \epsilon(x,y) \tag{2.18}$$

and

$$D(x,y) = q^{(x+y)/2} \sum_{j=0}^{(M/2)-1} \frac{1}{r_j}\left[R_{2j}(x) R_{2j+1}(y) - R_{2j+1}(x) R_{2j}(y)\right] \tag{2.19}$$

with

$$\Phi_j(x) = \sum_{y=0}^{\infty} \epsilon(y,x) q^{y/2} R_j(y). \tag{2.20}$$

2.4 Skew orthogonal polynomials

The explicit forms of polynomials with the skew orthogonality property (2.14) with respect to the skew product (2.15) is required. These can be calculated from general determinant formulas for skew orthogonal polynomials.

Lemma 1. *Let $\langle \, , \, \rangle$ be a general skew symmetric product. Let $\{R_j(x)\}_{j=0,1,\ldots}$ be the corresponding skew orthogonal polynomials which thus satisfy (2.14). Let $\{C_j(x)\}_{j=0,1,\ldots}$ be any family of monic polynomials. Assuming $\mathcal{D}_n$ and $\mathcal{E}_n$ as specified by (2.24) and (2.25) below are non-zero we have*

$$R_{2n}(x) = \mathcal{D}_n^{-1} \begin{vmatrix} C_{2n}(x) & J^{2n\ 2n-1} & J^{2n\ 2n-2} & \cdots & J^{2n\ 0} \\ C_{2n-1}(x) & J^{2n-1\ 2n-1} & J^{2n-1\ 2n-2} & \cdots & J^{2n-1\ 0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_0(x) & J^{0\ 2n-1} & J^{0\ 2n-2} & \cdots & J^{0\ 0} \end{vmatrix} \tag{2.21}$$

and

$$\begin{aligned} R_{2n+1}(x) \quad = \quad & \mathcal{E}_n^{-1} \begin{vmatrix} J^{2n+1\ 2n+1} & C_{2n+1}(x) & J^{2n+1\ 2n-1} & J^{2n+1\ 2n-2} & \cdots & J^{2n+1\ 0} \\ J^{2n\ 2n+1} & C_{2n}(x) & J^{2n\ 2n-1} & J^{2n\ 2n-2} & \cdots & J^{2n\ 0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ J^{0\ 2n+1} & C_0(x) & J^{0\ 2n-1} & J^{0\ 2n-2} & \cdots & J^{0\ 0} \end{vmatrix} \\ & + c_n R_{2n}(x), \end{aligned} \tag{2.22}$$

where

$$J^{mn} = \langle C_m(x), C_n(y) \rangle, \tag{2.23}$$

$$\mathcal{D}_n = \begin{vmatrix} J^{2n-1\ 2n-1} & J^{2n-1\ 2n-2} & \cdots & J^{2n-1\ 0} \\ J^{2n-2\ 2n-1} & J^{2n-2\ 2n-2} & \cdots & J^{2n-2\ 0} \\ \vdots & \vdots & \ddots & \vdots \\ J^{0\ 2n-1} & J^{0\ 2n-2} & \cdots & J^{0\ 0} \end{vmatrix}, \tag{2.24}$$

$$\mathcal{E}_n = - \begin{vmatrix} J^{2n\ 2n+1} & J^{2n\ 2n-1} & \cdots & J^{2n\ 0} \\ J^{2n-1\ 2n+1} & J^{2n-1\ 2n-1} & \cdots & J^{2n-1\ 0} \\ \vdots & \vdots & \ddots & \vdots \\ J^{0\ 2n+1} & J^{0\ 2n-1} & \cdots & J^{0\ 0} \end{vmatrix} \tag{2.25}$$

and the c_n are arbitrary constants.

Proof. We note the skew orthogonality properties (2.14) are equivalent to the requirements

$$\langle C_j(x), R_{2n}(y) \rangle = \langle C_j(x), R_{2n+1}(y) \rangle = 0, \qquad j \le 2n-1 \tag{2.26}$$

and

$$\langle C_{2n}(x), R_{2n+1}(y) \rangle \neq 0. \tag{2.27}$$

The property (2.26) is immediate from the structure of (2.21) and (2.22), while (2.27) follows from the assumption that $\mathcal{D}_{n+1}$ is non-zero. Furthermore, by inspection both (2.21) and (2.22) give monic polynomials. $\square$

The structure of (2.15) suggests choosing $\{C_j(x)\}_{j=0,1,\dots}$ as particular Meixner polynomials. We recall [21] the Meixner polynomials are defined as

$$M_n(x;c,q) = {}_2F_1\left(\begin{matrix} -n, -x \\ c \end{matrix} \,\middle|\, 1 - \frac{1}{q} \right) \tag{2.28}$$

and satisfy

$$\sum_{x=0}^{\infty} \frac{(c)_x}{x!} q^x M_m(x;c,q) M_n(x;c,q) = \frac{q^{-n} n!}{(c)_n (1-q)^c} \delta_{m,n} \tag{2.29}$$

with $(c)_n = \Gamma(c+n)/\Gamma(c)$. We define monic polynomials $C_n(x)$ as

$$C_n(x) = \frac{n! q^n}{(q-1)^n} M_n(x;1,q) = x^n + \cdots . \tag{2.30}$$

From (2.29) the orthogonality relation for the $C_n(x)$ is

$$\sum_{x=0}^{\infty} q^x C_m(x) C_n(x) = h_n \delta_{m,n}, \tag{2.31}$$

where

$$h_n = \frac{(n!)^2 q^n}{(1-q)^{2n+1}}. \tag{2.32}$$

Using the symmetry $M_n(x;\gamma,q) = M_x(n;\gamma,q)$, (2.29) also implies the completeness relation

$$\sum_{n=0}^{\infty} \frac{1}{h_n} C_n(x) C_n(y) = q^{-x} \delta_{x,y}, \qquad x, y \in \mathbb{Z}_{\geq 0}. \tag{2.33}$$

With $\{C_j(x)\}$ so specified we seek to compute (2.23).

Proposition 2. *For $m < n$,*

$$J^{mn} = -J^{nm} = \langle C_m(x), C_n(y) \rangle = a_m b_n, \tag{2.34}$$

where

$$\begin{aligned} a_n &= n! \frac{\sqrt{\alpha}}{\sqrt{\alpha} - \sqrt{q}} \left(\frac{\sqrt{q}(1 - \sqrt{\alpha q})}{(1-q)(\sqrt{\alpha} - \sqrt{q})} \right)^n, \\ b_n &= n! \frac{1}{1 - \sqrt{\alpha q}} \left(\frac{\sqrt{q}(\sqrt{\alpha} - \sqrt{q})}{(1-q)(1 - \sqrt{\alpha q})} \right)^n. \end{aligned} \tag{2.35}$$

Proof. Set

$$F(y;\alpha) = q^{-y/2} \alpha^{-y/2} \sum_{x=y+1}^{\infty} q^{x/2} \alpha^{x/2} C_n(x). \tag{2.36}$$

It follows from the fact that $C_n(x)$ is a polynomial of degree n in x that $F(y;\alpha)$ is a polynomial of degree n in y (thus y can therefore be regarded as a continuous variable). We can therefore write

$$F(y;\alpha) = \sum_{j=0}^{n} \kappa_{nj} C_j(y). \tag{2.37}$$

To calculate the κ_{nj} we first multiply both sides by $q^{y/2}\alpha^{y/2}$ and subtract the same equation with y replaced by $y-1$ to obtain

$$C_n(y) = -F(y;\alpha) + q^{-1/2}\alpha^{-1/2}F(y-1;\alpha). \tag{2.38}$$

To proceed further we note, as can be verified from the definitions (2.28) and (2.30), that

$$\begin{aligned} xC_n(x-1) &= C_{n+1}(x) - \frac{q}{q-1}(2n+1)C_n(x) + \frac{n^2q^2}{(q-1)^2}C_{n-1}(x), \\ xC_n(x) &= C_{n+1}(x) - \frac{nq+n+q}{q-1}C_n(x) + \frac{n^2q}{(q-1)^2}C_{n-1}(x) \end{aligned} \tag{2.39}$$

(the second of these is the three term recurrence, the general structure of which holds for all sequences of orthogonal polynomials). Thus with (2.37) substituted in (2.38), we can multiply both sides of the equation by y and use the equations to get an equation involving only the linearly independent functions $C_{n+1}(y), C_n(y), \dots, C_0(y)$ in the variable y. Equating coefficients of these functions shows

$$\kappa_{nn} = \frac{\sqrt{q}}{\sqrt{\alpha}-\sqrt{q}}, \quad \kappa_{nk} = \frac{n!}{k!}\frac{\sqrt{\alpha q}}{(\sqrt{\alpha}-\sqrt{q})^2}\left(\frac{\sqrt{q}(1-\sqrt{\alpha q})}{(1-q)(\sqrt{\alpha}-\sqrt{q})}\right)^{n-k-1}, \quad k<n. \tag{2.40}$$

We will now make use of (2.37) and (2.40) to evaluate (2.23). Substituting (2.37) in (2.36), multiplying both sides by $q^yC_m(y)$ and summing over y, making use of (2.31) on the right hand side, we see that

$$\sum_{y=0}^{\infty} q^{y/2}\alpha^{-y/2}C_m(y)\sum_{x=y+1}^{\infty} q^{x/2}\alpha^{x/2}C_n(x) = \begin{cases} h_m\kappa_{nm}, & m \le n, \\ 0, & m > n. \end{cases} \tag{2.41}$$

Recalling (2.15), and making use of the explicit formulas (2.32) and (2.40), (2.34) follows. $\square$

In general [11, Prop. 6], if the skew product has a factorization (2.34) for monic polynomials $C_n(x)$, $n=0,1,\dots$, the corresponding monic skew orthogonal polynomials $R_k(x)$ can be written as a series in $\{C_n(x)\}_{n=0,1,\dots,k}$ for explicit coefficients involving the a_j's and b_j's, while $r_n = a_{2n}b_{2n+1}$. These facts can be seen from the determinant formulas (2.21) and (2.22). We thus have the following result.

Proposition 3. *Let $C_n(x)$ be specified by (2.30). The monic skew orthogonal polynomials with respect to (2.15) are given in terms of these polynomials by*

$$\begin{aligned} R_{2n}(x) &= C_{2n}(x) + \sum_{k=0}^{n-1}\frac{(2n)!}{(2k)!}\frac{q^{n-k}}{(1-q)^{2n-2k}}C_{2k}(x) \\ &\quad -\frac{\sqrt{\alpha}-\sqrt{q}}{1-\sqrt{\alpha q}}\sum_{k=0}^{n-1}\frac{(2n)!}{(2k+1)!}\frac{q^{n-k-(1/2)}}{(1-q)^{2n-2k-1}}C_{2k+1}(x), \end{aligned} \tag{2.42}$$

$$R_{2n+1}(x) = C_{2n+1}(x) - \frac{\sqrt{\alpha}-\sqrt{q}}{1-\sqrt{\alpha q}}\frac{\sqrt{q}}{1-q}(2n+1)C_{2n}(x), \tag{2.43}$$

while the corresponding normalization has the explicit value

$$r_n = \frac{(2n)!(2n+1)!\sqrt{\alpha}q^{2n+(1/2)}}{(1-q)^{4n+1}(1-\sqrt{\alpha q})^2}. \tag{2.44}$$

Proof. These results follow from the general formulas of [11]. The only point which requires clarification is that we have chosen for the arbitrary constant c_n in (2.22)

$$c_n = -\frac{\sqrt{\alpha}-\sqrt{q}}{1-\sqrt{\alpha q}}\frac{\sqrt{q}}{1-q}(2n+1),$$

which gives the simplest expression for $R_{2n+1}(x)$ in the basis $\{C_j(x)\}$. □

We note that following the procedure used in [13], the triangular structures (2.42) and (2.43) can be inverted to give

$$\begin{aligned} C_{2n}(x) = R_{2n}(x) + \frac{\gamma^{2n}(2n)!q^n}{(1-q)^{2n}} \\ \times\left[\left(1-\frac{1}{\gamma^2}\right)\sum_{k=0}^{n-1}\frac{(1-q)^{2k}}{\gamma^{2k}(2k)!q^k}R_{2k}(x) + \sum_{k=0}^{n-1}\frac{(1-q)^{2k+1}}{\gamma^{2k+1}(2k+1)!q^{k+(1/2)}}R_{2k+1}(x)\right] \end{aligned} \tag{2.45}$$

and

$$\begin{aligned} C_{2n+1}(x) &= \gamma\frac{q^{1/2}}{1-q}(2n+1)R_{2n}(x) + \frac{\gamma^{2n+1}(2n+1)!q^{n+(1/2)}}{(1-q)^{2n+1}} \\ &\quad\times\left[\left(1-\frac{1}{\gamma^2}\right)\sum_{k=0}^{n-1}\frac{(1-q)^{2k}}{\gamma^{2k}(2k)!q^k}R_{2k}(x) + \sum_{k=0}^{n}\frac{(1-q)^{2k+1}}{\gamma^{2k+1}(2k+1)!q^{k+(1/2)}}R_{2k+1}(x)\right], \end{aligned} \tag{2.46}$$

where

$$\gamma = \frac{\sqrt{\alpha}-\sqrt{q}}{1-\sqrt{\alpha q}}. \tag{2.47}$$

2.5 A summation formula

Let us write

$$C_n(x) = \sum_{j=0}^{n}\beta_{nj}R_j(x), \quad \beta_{nn} = 1. \tag{2.48}$$

We see from the explicit formulas (2.45), (2.46) that the coefficients have the factorization property

$$\beta_{nj} = \alpha_n\gamma_j \tag{2.49}$$

for certain α_n, γ_j. According to [23, Eq. (3.42)], under such a circumstance the matrix element $S(x,y)$ can be summed.

Proposition 4. *The expression (2.17) can be simplified to read*

$$\begin{aligned} S(x,y) &= q^{(x+y)/2}\frac{1}{h_{M-1}}\frac{C_M(x)C_{M-1}(y)-C_{M-1}(x)C_M(y)}{x-y} + q^{y/2}\frac{1}{r_{(M-2)/2}}\frac{1-q}{M\gamma q^{1/2}} \\ &\quad\times\left(\Phi_{M-2}(x) - q^{x/2}\frac{r_{(M-2)/2}}{h_{M-1}}C_{M-1}(x)\right)(C_M(y)-R_M(y)). \end{aligned} \tag{2.50}$$

Proof. We know from general formulas [23] that with $\{\beta_{nj}\}_{j=0,\dots,n}$ the lower triangular transition matrix for the change of variables from the monic skew orthogonal polynomials $\{R_j(x)\}_{j=0,1,\dots}$ to the monic orthogonal polynomials $\{C_j(x)\}_{j=0,1,\dots}$ as implied by (2.48), the quantity (2.20) permits the expansions

$$\begin{aligned}\Phi_{2k-1}(x) &= -q^{x/2}\sum_{\nu=2k-2}^{\infty}\frac{C_\nu(x)}{h_\nu}\beta_{\nu\ 2k-2}\,r_{k-1},\\ \Phi_{2k-2}(x) &= q^{x/2}\sum_{\nu=2k-1}^{\infty}\frac{C_\nu(x)}{h_\nu}\beta_{\nu\ 2k-1}\,r_{k-1}.\end{aligned}\tag{2.51}$$

Further, we know that these formulas substituted into (2.17) give

$$S(x,y) = q^{(x+y)/2}\sum_{\nu=0}^{M-1}\frac{C_\nu(x)C_\nu(y)}{h_\nu} + q^{(x+y)/2}\sum_{\nu=M}^{\infty}\sum_{k=0}^{M-1}\frac{C_\nu(x)}{h_\nu}\beta_{\nu k}R_k(y).\tag{2.52}$$

According to the Christoffel-Darboux formula from the theory of orthogonal polynomials, the first summation gives the first term on the right hand side of (2.50). Moreover, making use of the factorization (2.49) shows

$$\sum_{k=0}^{M-1}\beta_{\nu k}R_k(y) = \frac{\gamma^{\nu-M}q^{(\nu-M)/2}}{(1-q)^{\nu-M}}\frac{\nu!}{M!}\left(C_M(y)-R_M(y)\right).$$

After further use of (2.51) the second term of (2.50) results. □

3 Poissonized Random Involutions

3.1 Poissonization

Essential to the rationale underlying the introduction of the geometrical model in §2.2 is its reduction in an appropriate limit to the Poissonization of random involutions. To see how this comes about, the procedure for the Poissonization of random involutions given in [2] must be revised.

Consider involutions of $\{1,2,\dots,N\}$, and specify the number of fixed points therein to be m. Catalogue the involutions according to $(\lambda_N^{(k)} =: \lambda_k)_{k=1,\dots,N}$ where $\lambda_N^{(k)}$ is specified by (1.1). The Robinson-Schensted-Knuth correspondence (see e.g. [15]) tells us that there is a bijection between all involutions with a given value of $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_N)$ and standard tableaux of shape λ and content N. Furthermore, the number of fixed points m is equal to $\sum_{j=1}^N(-1)^{j-1}\lambda_j$, and so restricts the permissible λ.

Let the number of standard tableaux of shape λ and content N be denoted f_λ. With $\ell(\lambda)$ denoting the number of non-zero parts of λ, and

$$V_p(\lambda) := \prod_{1\le j<l\le p}(\lambda_j-\lambda_l+l-j), \qquad W_p(\lambda) = \prod_{j=1}^{p}\frac{1}{(\lambda_j+p-j)!},\tag{3.1}$$

it is known that [15]

$$f_\lambda = N!V_{l(\lambda)}(\lambda)W_{l(\lambda)}.\tag{3.2}$$

Further, let $t_{n,m}$ denote the number of involutions of $\{1,2,\dots,N\}$ with m fixed points and n 2-cycles. We have

$$t_{n,m} = \sum_{\lambda \in \Omega^{(\infty)}_{2n+m,m}} f_\lambda = \frac{(2n+m)!}{2^n n! m!}, \tag{3.3}$$

where $\Omega^{(\infty)}_{2n+m,m}$ is the set of partitions with

$$\sum_{j=1}^{\ell(\lambda)} \lambda_j := |\lambda| = 2n+m, \qquad \sum_{j=1}^{\ell(\lambda)} (-1)^{j-1}\lambda_j = m. \tag{3.4}$$

In terms of these quantities the probability that such an involution corresponds to a standard tableau of shape λ is then given by

$$\mathbb{Q}_{n,m}(\lambda) = \frac{f_\lambda}{t_{n,m}} \chi_{\lambda \in \Omega^{(\infty)}_{2n+m,m}}.$$

And the Poissonized form of this, in which n and m are regarded as random variables from distinct Poisson distributions, is

$$\begin{aligned} \mathbb{Q}^{(z_1,z_2)}(\lambda) &:= \frac{e^{-z_1-z_2} z_1^n z_2^m}{n! m!} \frac{1}{t_{n,m}} f_\lambda \\ &= e^{-z_1-z_2} z_1^n z_2^m \frac{2^n}{(2n+m)!} f_\lambda, \end{aligned} \tag{3.5}$$

where the second equality follows upon using (3.3), and n, m are related to λ by (3.4).

For future applications it is convenient to write this in terms of (N,m) rather than (n,m). Setting

$$z_1 = Q/2, \qquad z_2 = \sqrt{Q\alpha} \tag{3.6}$$

and with $|\lambda| = N$ defining

$$\mathbb{P}^{(Q,\alpha)}(\lambda) = e^{-\sqrt{\alpha Q} - Q/2} \frac{\alpha^{m/2} Q^{N/2}}{N!} f_\lambda, \tag{3.7}$$

we see that

$$\mathbb{Q}^{(Q/2,(\alpha Q)^{1/2})}(\lambda) = \mathbb{P}^{(Q,\alpha)}(\lambda). \tag{3.8}$$

Also for future application, we note from the fact that $\mathbb{P}^{(Q,\alpha)}$ is a probability distribution that one has the summation

$$\sum_{N=0}^{\infty} \frac{Q^{N/2}}{N!} t_N^{(\alpha)} = e^{\sqrt{\alpha Q} + Q/2}, \tag{3.9}$$

where

$$t_N^{(\alpha)} := \sum_{\lambda : |\lambda| = N} t_{n,m} \alpha^{m/2} \tag{3.10}$$

(the case $\alpha = 1$ is given in [20, §5.1.4, eq. (42)]).

Consider now a function $g(\lambda)$ of the parts of a partition λ. A concrete example of future use is

$$g(\lambda) = \prod_{j=1}^{\infty} (1 + u(\lambda_j - j)), \tag{3.11}$$

where $u(x)$ is any function which vanishes for $x<0$. For such functions we define the Poissonized average as

$$\langle g(\lambda)\rangle^{(Q,\alpha)} = \sum_\lambda g(\lambda)\mathbb{P}^{(Q,\alpha)}(\lambda). \tag{3.12}$$

We want to show that this average results as a limit of the model of §2.2.

Proposition 5. *Let $g(\lambda)$ be a function of the parts of a partition λ which furthermore satisfies the bound*

$$|g(\lambda)| \le c^{\sum_{j=1}^{\ell(\lambda)} \lambda_j} \tag{3.13}$$

for some $c>0$. With $P_M(\lambda;q)$ given by (2.2) we have

$$\lim_{M\to\infty} \sum_{\lambda:\ell(\lambda)\le M} g(\lambda) P_M(\lambda; Q/M^2) = \langle g(\lambda)\rangle^{(Q,\alpha)}. \tag{3.14}$$

Proof. We essentially follow Johansson [19], who proved the analogous result for Poissonized permutations. Using the explicit formula (2.2), we see

$$\begin{aligned}\sum_{\lambda:\ell(\lambda)\le M} g(\lambda)P_M(\lambda;q) &= (1-\sqrt{\alpha q})^M (1-q)^{M(M-1)/2} \sum_{\lambda:\ell(\lambda)\le M} g(\lambda) \prod_{j=1}^M q^{\lambda_j/2} \prod_{j=1}^M \alpha^{\sum_{j=1}^M (-1)^{j-1}\lambda_j} \\ &\quad \times \prod_{1\le j<k\le M} \frac{\lambda_j-\lambda_l+l-j}{l-j} \\ &= (1-\sqrt{\alpha q})^M (1-q)^{M(M-1)/2} \sum_{\lambda:\ell(\lambda)\le M} q^{N/2}\alpha^{m/2} g(\lambda) V_M(\lambda) \prod_{j=1}^{M-1} \frac{1}{j!}.\end{aligned} \tag{3.15}$$

With $\Omega_{N,m}^{(M)}$ denoting the set of partitions λ with the properties (3.4) but constrained so that $\ell(\lambda)\le M$, recalling (3.1) we see

$$\begin{aligned}\sum_{\lambda\in\Omega_{N,m}^{(M)}} g(\lambda)V_M(\lambda)\prod_{j=1}^{M-1}\frac{1}{j!} &= \sum_{\lambda\in\Omega_{N,m}^{(M)}} g(\lambda)\frac{V_{l(\lambda)}(\lambda)W_{l(\lambda)}(\lambda)}{W_M(\lambda)} \prod_{j=1}^{M-1}\frac{1}{j!} \\ &= \sum_{\lambda\in\Omega_{N,m}^{(\infty)}} g(\lambda)\frac{f_\lambda}{N!}\prod_{j=1}^M \frac{(\lambda_j+M-j)!}{(M-j)!}.\end{aligned} \tag{3.16}$$

From the assumed bound (3.13) and the further bound

$$\frac{(\lambda_j+M-j)!}{M^{\lambda_j}(M-j)!} \le 1$$

(a consequence of Stirling's formula), we see from (3.16) that with q replaced by Q/M^2 in (3.15), the sum over $\lambda:\ell(\lambda)\le M$ for $M\to\infty$ in the latter is itself bounded by

$$\sum_\lambda \frac{\tilde{Q}^{N/2}\alpha^{m/2}}{N!} t_{N,m}, \qquad \tilde{Q} = Qc^2.$$

But according to (3.10) and (3.9) this is summed as the r.h.s. of (3.9) with Q replaced by $\tilde{Q}$. Thus we have a uniform bound in N for the sum in (3.15) with q replaced by Q/M^2. The limit $M \to \infty$ can therefore be taken term-by-term. Doing so gives (3.14). $\square$

We remark that (3.12) can be written

$$\langle g(\lambda) \rangle^{(Q,\alpha)} = e^{-\sqrt{\alpha Q} - Q/2} \sum_{N=0}^{\infty} \frac{Q^{N/2}}{N!} t_N^{(\alpha)} \langle g(\lambda) \rangle_N^{(\alpha)}, \tag{3.17}$$

$$\langle g(\lambda) \rangle_N^{(\alpha)} := \frac{1}{t_N^{(\alpha)}} \sum_{\lambda : |\lambda| = N} g(\lambda) \alpha^{m/2} f_\lambda. \tag{3.18}$$

Some insight into the role of α can be obtained by computing the mean number of fixed points in the ensemble (3.18). From (3.12), the identity (3.8), the constraint involving m in (3.4), and the explicit form (3.5) it is immediate that

$$\Big\langle \sum_{j=1}^{\ell(\lambda)} (-1)^{j-1} \lambda_j \Big\rangle^{(Q,\alpha)} = \sqrt{\alpha Q}.$$

This in (3.17) and use of (3.9) implies

$$\Big\langle \sum_{j=1}^{\ell(\lambda)} (-1)^{j-1} \lambda_j \Big\rangle_N^{(\alpha)} = \sqrt{\alpha} \frac{t_{N-1}^{(\alpha)}}{t_N^{(\alpha)}} N.$$

Recalling the explicit formula (3.3), the sum (3.10) can be estimated to give $t_N^{(\alpha)} / t_{N-1}^{(\alpha)} \sim \sqrt{N}$ as $N \to \infty$ ($\alpha > 0$). Hence

$$\Big\langle \sum_{j=1}^{\ell(\lambda)} (-1)^{j-1} \lambda_j \Big\rangle_N^{(\alpha)} \underset{N \to \infty}{\sim} \sqrt{\alpha N},$$

which tells us that in the ensemble (3.18) the mean number of fixed points is necessarily proportional to $\sqrt{N}$, with proportionality constant $\sqrt{\alpha}$.

3.2 A Poissonized average and limiting correlations

According to the equations (2.7)–(2.9), to calculate the distribution function $\Pr(h_1 \le a_1, \dots, h_l \le a_l)$, it is sufficient to compute the average

$$\Big\langle \prod_{l=1}^{M} \Big(1 - \sum_{r=1}^{k} \xi_r \chi_{I_r}^{(l)} \Big) \Big\rangle_{\rm Psym}. \tag{3.19}$$

Let us suppose each end point a_j in the $\{I_r\}$ is a positive integer when measured from M. We indicate this by the replacement $I_r \mapsto M + I_r$. The quantity being averaged in (3.19) is now of the form (3.11), and (3.14) tells us that

$$\lim_{M \to \infty} \Big\langle \prod_{l=1}^{M} \Big(1 - \sum_{r=1}^{k} \xi_r \chi_{I_r}^{(l)} \Big) \Big\rangle_{\rm Psym} \Big|_{q = Q/M^2} = \Big\langle \prod_{l=1}^{\infty} \Big(1 - \sum_{r=1}^{k} \xi_r \chi_{\lambda_l - l \in I_r} \Big) \Big\rangle^{(\alpha,Q)}. \tag{3.20}$$

We know from the proof of Proposition 5 that the convergence is uniform in M, and can therefore be taken term-by-term in (2.9) to give

$$\Big\langle \prod_{l=1}^{\infty} \Big(1 - \sum_{r=1}^{k} \xi_r \chi_{\lambda_l - l \in I_r}\Big)\Big\rangle^{(\alpha,Q)} \\ = 1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \sum_{h_1,\dots,h_p=0}^{\infty} \prod_{i=1}^{p} \Big(\sum_{r=1}^{k} \xi_r \chi_{I_r}^{(i)} \Big) \lim_{M\to\infty} \rho_{(p)}(h_1+M,\dots,h_p+M)\Big|_{q=Q/M^2}. \tag{3.21}$$

Moreover, the uniform convergence tells us that the limit $M \to \infty$ can be taken term-by-term in the summations of the quantities specifying $\rho_{(k)}$ (recall (2.17)–(2.20)). To calculate these limits requires an appropriate asymptotic formula relating to $C_n(x)$.

Proposition 6. *For $M \to \infty$ we have*

$$C_n(x+M)\Big|_{q=Q/M^2} \sim \frac{n! q^{(n-M-x)/2}}{(1-q)^n} J_{-n+M+x}(2\sqrt{Q}) \tag{3.22}$$

uniformly in $n \in \mathbb{Z}_{\geq 0}$.

Proof. Following [18], we use the generating function of the Meixner polynomials [21]

$$\left(1-\frac{t}{q}\right)^x (1-t)^{-x-c} = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} M_n(x;c,q) t^n, \tag{3.23}$$

to derive the integral representation

$$M_n(x;c,q) = \frac{n!}{2\pi i (c)_n} \oint \mathrm{d}t \; t^{-n-1} \left(1-\frac{t}{q}\right)^x (1-t)^{-x-c}, \tag{3.24}$$

where the path of integration encloses the origin anticlockwise. It follows from (3.24) that

$$\begin{aligned} & C_n(x+M)|_{q=Q/M^2} \\ = \; & \frac{n! q^n}{2\pi i (q-1)^n} \oint \mathrm{d}t \; t^{-n-1} \left(1-\frac{t}{q}\right)^{x+N} (1-t)^{-x-N-1} \\ = \; & \frac{n! q^{(n-M-x)/2}}{(1-q)^n} \frac{(-1)^{-n+M+x}}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; z^{-n+M+x} \left(1-\frac{\sqrt{Q}}{Mz}\right)^{x+M} \left(1-\frac{\sqrt{Q}z}{M}\right)^{-x-M-1} \end{aligned} \tag{3.25}$$

with $z = re^{i\theta}$, $r > 0$. For large M the integral has the leading form

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \;\; z^{-n+M+x} \exp\left\{ \sqrt{Q}\left(z - \frac{1}{z}\right)\right\} = (-1)^{-n+M+x} 2\pi J_{-n+M+x}(2\sqrt{Q})$$

uniform in $n \in \mathbb{Z}$, where $J_k(x)$ denotes the Bessel function, and the result follows. □

Asymptotic formulas for $\{R_j(x)\}$, $\{\Phi_j(x)\}$ appearing in the matrix elements of (2.16) can now be obtained.

Corollary 2. *Let M be even. We have*

$$R_{M-2n}(x+M)\Big|_{q=Q/M^2} \sim (M-2n)!q^{-(2n+x)/2}\Big(\sum_{l=0}^{M/2-n} J_{2n+2l+x}(2\sqrt{Q}) - \sqrt{\alpha}\sum_{l=1}^{M/2-n} J_{2n+2l-1+x}(2\sqrt{Q})\Big), \tag{3.26}$$

$$R_{M-2n+1}(x+M)\Big|_{q=Q/M^2} \sim (M-2n+1)!q^{-(2n+x-1)/2}\Big(J_{2n-1+x}(2\sqrt{Q}) - \sqrt{\alpha}J_{2n+x}(2\sqrt{Q})\Big). \tag{3.27}$$

$$\Phi_{M-2n}(x+M)\Big|_{q=Q/M^2} \sim \alpha^n(M-2n)!q^{M-2n)/2}\sum_{\nu=-2n+1}^{\infty}\alpha^{\nu/2}J_{-\nu+x}(2\sqrt{Q}), \tag{3.28}$$

$$\Phi_{M-2n+1}(x+M)\Big|_{q=Q/N^2} \sim \alpha^{n+1/2}(M-2n+1)!q^{M-2n+1)/2} \times \sum_{\nu=-2n}^{\infty}\alpha^{\nu/2}\Big(-J_{-\nu+x}(2\sqrt{Q}) + J_{\nu-2+x}(2\sqrt{Q})\Big). \tag{3.29}$$

Proof. The expansions (3.26) and (3.27) follow immediately upon use of (3.22) in (2.42) and (2.43). In relation to $\{\Phi_j(x)\}$ we read off explicit formulas for the β_{nj} as defined in (2.48) from (2.45), (2.46) and substitute in (2.51) to obtain

$$\begin{aligned}
\Phi_{2n}(x) &= q^{x/2}\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha q})^2}\frac{(2n)!q^n}{\gamma^{2n+1}(1-q)^{2n}}\sum_{\nu=2n+1}^{\infty}\frac{(1-q)^{\nu+1}\gamma^{\nu}}{\nu!q^{\nu/2}}C_\nu(x),\\
\Phi_{2n+1}(x) &= q^{x/2}\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha q})^2}\frac{(2n+1)!q^{n+(1/2)}}{\gamma^{2n}(1-q)^{2n+1}}\\
&\quad\times\sum_{\nu=2n}^{\infty}\frac{(1-q)^{\nu+1}\gamma^{\nu}}{\nu!q^{\nu/2}}\left(-C_\nu(x)+\frac{(1-q)^2}{(\nu+2)(\nu+1)q}C_{\nu+2}(x)\right).
\end{aligned} \tag{3.30}$$

Use of (3.22) now gives (3.28), (3.29). □

Proposition 7. *We have*

$$\begin{aligned}
\rho_{(k)}^{(Q,\alpha)}(h_1,\dots,h_k) &:= \lim_{M\to\infty}\rho_{(k)}(h_1+M,\dots,h_k+M)\Big|_{q=Q/M^2}\\
&= \mathrm{qdet}\begin{bmatrix}\bar{S}(h_j,h_l) & \bar{I}(h_j,h_l)\\ \bar{D}(h_j,h_l) & \bar{S}(h_l,h_j)\end{bmatrix}_{j,l=1,\dots,k},
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
\bar{S}(x,y) &= \frac{\sqrt{Q}}{x-y}\Big(J_x(2\sqrt{Q})J_{y+1}(2\sqrt{Q}) - J_y(2\sqrt{Q})J_{x+1}(2\sqrt{Q})\Big)\\
&\quad - \sum_{j=0}^{\infty}\alpha^{j/2}J_{-j+x}(2\sqrt{Q})\sum_{l=0}^{\infty}\Big(J_{2l+2+y}(2\sqrt{Q}) - \sqrt{\alpha}J_{2l+1+y}(2\sqrt{Q})\Big),
\end{aligned} \tag{3.32}$$

$$\bar{I}(x,y) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\alpha^{k/2}$$
$$\Big(J_{n+x}(2\sqrt{Q})J_{n-k+y}(2\sqrt{Q}) - J_{n+y}(2\sqrt{Q})J_{n-k+x}(2\sqrt{Q})\Big) + \epsilon(x,y), \tag{3.33}$$

$$\bar{D}(x,y)$$
$$= \frac{1}{\sqrt{\alpha}}\sum_{l=1}^{\infty}\sum_{j=1}^{l}\Big(J_{2l+x}(2\sqrt{Q}) - \sqrt{\alpha}J_{2l+x+1}(2\sqrt{Q})\Big)\Big(J_{2j+y-1}(2\sqrt{Q}) - \sqrt{\alpha}J_{2j+y}(2\sqrt{Q})\Big)$$
$$-(x\leftrightarrow y). \tag{3.34}$$

Proof. Using the results of Corollary 2, from the form of the matrix elements (2.17)–(2.20) we compute

$$\bar{S}(x,y) = \lim_{M\to\infty} S(x+M, y+M)\Big|_{q=Q/M^2}$$

and similarly for $\bar{I}(x,y)$, $\bar{D}(x,y)$. □

We note that Bessel function identities can be used to verify that the first term in (3.32) can be written in a denominator free from [7, Prop. 2.9]. This allows (3.32) to be replaced by

$$\bar{S}(x,y) = \sum_{n=1}^{\infty} J_{n+x}(2\sqrt{Q})J_{n+y}(2\sqrt{Q})$$
$$-\sum_{j=0}^{\infty}\alpha^{j/2}J_{-j+x}(2\sqrt{Q})\sum_{l=0}^{\infty}\Big(J_{2l+2+y}(2\sqrt{Q}) - \sqrt{\alpha}J_{2l+1+y}(2\sqrt{Q})\Big). \tag{3.35}$$

□

The correlations (3.31) relate to the measure on diagrams of partitions (3.7). A closely related measure, introduced in [2], is

$$\tilde{\mathbb{P}}^{(Q,\beta)}(\lambda) = e^{-\sqrt{\beta Q}-Q/2}\frac{\beta^{m/2}Q^{N/2}}{N!}f_\lambda, \tag{3.36}$$

where $N = |\lambda|$ and $m = \sum_{j=1}^{\ell(\lambda)}(-1)^{j-1}\lambda'_j$ with λ'_j denoting the length of the jth column of the diagram of λ. This relates to decreasing subsequences in the involution. In the case $\beta = 0$ only rows of even length are permitted. For this model the corresponding k-point correlations have recently been computed by Ferrari [9, Lemma 5.2], in the context of the polynuclear growth model from a flat substrate. The results obtained have a very similar structure to that exhibited in Proposition 7. For example, the density (one-point correlation) is computed for (3.36) with $\beta = 0$ as

$$\tilde{\rho}_{(1)}^{(Q,0)}(x) = \sum_{n=1}^{\infty}\Big(J_{n+x}(2\sqrt{Q})\Big)^2 - J_{x+1}(2\sqrt{Q})\Big(\sum_{m=1}^{\infty}J_{x+2m-1}(2\sqrt{Q}) - \frac{(1-(-1)^x)}{2}\Big), \tag{3.37}$$

while for (3.7) with $\alpha = 0$ Proposition 7 and (3.35) give

$$\rho_{(1)}^{(Q,0)}(x) = \sum_{n=1}^{\infty}\Big(J_{n+x}(2\sqrt{Q})\Big)^2 - J_x(2\sqrt{Q})\sum_{m=1}^{\infty}J_{x+2m}(2\sqrt{Q}). \tag{3.38}$$

We also make mention that in the works [5, 11, 16] the parameter dependent correlations have recently been calculated for other models generalizing the geometric picture of random involutions.

4 Asymptotic Correlation Functions

4.1 De-Poissonization

Following the pioneering work [17], the following de-Poissonization lemma was derived in [3].

Lemma 8. *Let $d > 0$ be an arbitrary positive real number, and set*

$$\begin{aligned} \mu_n^{(d)} &= n + (2\sqrt{d+1}+1)\sqrt{n \log n}, \\ \nu_n^{(d)} &= n - (2\sqrt{d+1}+1)\sqrt{n \log n}. \end{aligned}$$

Let $q = \{q_{n_1,n_2}\}_{n_1,n_2 \geq 0}$ be such that

$$q_{n_1,n_2} \geq q_{n_1+1,n_2}, \quad q_{n_1,n_2} \geq q_{n_1,n_2+1}, \quad 0 \leq q_{n_1,n_2} \leq 1$$

and define

$$\phi(\lambda_1, \lambda_2) = e^{-\lambda_1-\lambda_2} \sum_{n_1,n_2 \geq 0} q_{n_1,n_2} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1! n_2!}.$$

Then there exists constants C and n_0 such that for all $n_1, n_2 \geq n_0$

$$\phi(\mu_{n_1}^{(d)}, \mu_{n_2}^{(d)}) - C(n_1^{-d} + n_2^{-d}) \leq q_{n_1,n_2} \leq \phi(\nu_{n_1}^{(d)}, \nu_{n_2}^{(d)}) + C(n_1^{-d} + n_2^{-d}). \tag{4.1}$$

In using this lemma, we take $q_{n,m}$ as the joint distribution of $\{\lambda_{n,m}^{(j)}\}_{j=1,\dots,l}$ for random involutions of $\{1,\dots,N\}$, $N = 2n+m$ with n 2-cycles and m fixed points, and thus

$$q_{n,m} = \Pr(\lambda_{n,m}^{(1)} \leq a_1, \dots, \lambda_{n,m}^{(l)} \leq a_l).$$

We know from the argument of [3, Lemma 7.5] that this satisfies the monotonicity conditions required for the validity of Lemma 8. We further choose

$$a_j = 2\sqrt{Q} + Q^{1/6} s_j, \qquad n_1 = [Q/2], \qquad n_2 = [\sqrt{Q} - 2wQ^{1/3}].$$

The lemma then gives

$$\begin{aligned} &\lim_{Q\to\infty} e^{-z_1-z_2} \sum_{n,m\geq 0} \frac{z_1^n z_2^m}{n!m!} \Pr(\lambda_{n,m}^{(1)} \leq a_1, \dots, \lambda_{n,m}^{(l)} \leq a_l)\Big|_{\substack{a_j = 2\sqrt{Q}+Q^{1/6}s_j \\ z_1 = Q/2 \\ z_2 = \sqrt{Q}-2wQ^{1/3}}} \\ &= \lim_{N\to\infty} \Pr(\lambda_{n,m}^{(1)} \leq a_1, \dots, \lambda_{n,m}^{(l)} \leq a_l)\Big|_{\substack{a_j = 2\sqrt{Q}+Q^{1/6}s_j \\ n = [Q/2] \\ m = [\sqrt{Q}-2wQ^{1/3}]}}, \end{aligned} \tag{4.2}$$

assuming the limit on the left hand side exists. Regarding the left hand side, we know from (3.8) that it is equal to

$$\lim_{Q\to\infty} \Pr^{(Q,\alpha)}(\lambda_1 \leq a_1, \dots, \lambda_l \leq a_l)\Big|_{\substack{a_j = 2\sqrt{Q}+Q^{1/6}s_j \\ \sqrt{\alpha} = 1-2w/Q^{1/6}}}, \tag{4.3}$$

where $\Pr^{(Q,\alpha)}$ refers to the ensemble defined by the measure (3.7).

As demonstrated in (2.7)–(2.9), the probability in (4.3) is fully determined by the right hand side of (3.20). Using (3.20) and (3.21) this in turn can be written

$$\begin{aligned}\Big\langle \prod_{l=1}^{\infty}\Big(1-\sum_{r=1}^{k}\xi_r\chi_{\lambda_l - l\in I_r}\Big)\Big\rangle^{(\alpha,Q)} \\ = 1+\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}\sum_{h_1,\dots,h_p=0}^{\infty}\prod_{i=1}^{p}\Big(\sum_{r=1}^{k}\xi_r\chi_{I_r}^{(i)}\Big)\rho_{(p)}^{(Q,\alpha)}(h_1,\dots,h_p).\end{aligned} \tag{4.4}$$

Therefore our task is to compute the $Q\to\infty$ limit of this quantity with

$$I_r=(a_r,a_{r-1}),\quad a_0=\infty, a_j=2\sqrt{Q}+Q^{1/6}s_j\ (j=1,\dots,k),\quad \sqrt{\alpha}=1-2w/Q^{1/6}. \tag{4.5}$$

4.2 The limit $Q\to\infty$

We will make use of a lemma due to Soshnikov [28, Lemma 2], in a form close to that used in [6].

Proposition 9. *Consider a sequence of point processes labelled by a parameter L. Suppose that for $L\to\infty$, and after the linear scaling $x_j\mapsto A_L(x_j)=\alpha_L+a_Lx_j$ of each of the coordinates, the sequence approaches a limit point process with correlations $\{\rho_k\}_{k=1,2,\dots}$ such that*

$$\int_{-\infty}^{\infty}dx_1\dots\int_{-\infty}^{\infty}dx_k\prod_{i=1}^{k}\Big(\sum_{r=1}^{l}\xi_r\chi_{I_r}^{(i)}\Big)\rho_{(k)}(x_1,\dots,x_k)=o(k!), \tag{4.6}$$

and suppose furthermore that for each $r=1,\dots,l$

$$\lim_{L\to\infty}a_L^k\int_{A_L(I_r)}dx_1\cdots\int_{A_L(I_r)}dx_k\,\rho_{(k)}^{(L)}(A_L(x_1),\dots,A_L(x_k))=\int_{I_r}dx_1\cdots\int_{I_r}dx_k\,\rho_{(k)}(x_1,\dots,x_k). \tag{4.7}$$

One then has

$$\begin{aligned}\lim_{L\to\infty}\Big\langle\prod_i\Big(1-\sum_{r=1}^{l}\xi_r\chi_{A_L(I_r)}^{(i)}\Big)\Big\rangle^{(L)} \\ =1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_k\prod_{i=1}^{k}\Big(\sum_{r=1}^{l}\xi_r\chi_{I_r}^{(i)}\Big)\rho_{(k)}(x_1,\dots,x_k).\end{aligned} \tag{4.8}$$

The essential idea behind this result is that the condition (4.6) implies the moment problem for the number of particles in $\{I_r\}$ is definite, and thus the convergence of moments, which are integrals of correlation functions, implies the convergence of distributions.

In using Proposition 9, the sequence of point processes will be those specified by the correlations (3.31) and thus labelled by the parameter Q. The limiting point process is that specified by the correlations appearing on the r.h.s. of (4.7). These correlations we will calculate to be (1.13). To deduce (4.8) we must first verify (4.6) for the correlations (1.13).

Lemma 10. *There exists a positive number M such that*

$$\rho_{(k)}^{\rm scaled}(x_1,\dots,x_k)\le e^{-(x_1+\dots+x_k)}k^{k/2}M^k \tag{4.9}$$

for all $x_i\in[s_0,\infty)$ $(i=1,\dots,k)$, were s_0 is arbitrary but fixed.

Proof. For $X \to \infty$ we know that $\mathrm{Ai}(X) = O(e^{-2X^{3/2}/3})$. Recalling (1.3), we see from the explicit formulas (1.14) that

$$
\begin{aligned}
&f^{22}(X,Y) = f^{11}(Y,X) = O(e^{-2Y^{3/2}/3})O(e^{uX/2}), \quad f^{21}(X,Y) = O(e^{uX/2})O(e^{uY/2}),\\
&f^{12}(X,Y) = O(e^{-2X^{3/2}/3})O(e^{-2Y^{3/2}/3}).
\end{aligned}
\tag{4.10}
$$

The qdet can be expanded [8]

$$
\mathrm{qdet}\,[f(X_i,X_j)]_{i,j=1,\dots,k} = \sum_{\substack{\text{disjoint cycles}\\ \in S_k}} (-1)^{k-l} \prod_{1}^{l} \Big(f(X_a,X_b) f(X_b,X_c) \cdots f(X_d,X_a) \Big)^{(0)}, \tag{4.11}
$$

where the superscript (0) denotes the operation (1/2)Tr, and l denotes the number of disjoint cycles. We see from this that each term consists of factors of the form $f^{11}(X,X)$, $f^{22}(X,X)$, $f^{11}(X,Y)f^{22}(Y,X')$, or $f^{12}(X,Y)f^{21}(Y,X')$. The bounds (4.10) tell us that each term is bounded by $O(e^{-(x_1+\cdots+x_k)})$ (of course a sharper bound can be given, but this is sufficient for our purpose).

We must also bound the dependence on k. For this we note that in general the replacement of the matrix elements of (1.13) by

$$
\begin{aligned}
&f^{11}(X,Y) \mapsto \frac{a(X)}{a(Y)} f^{11}(X,Y), \quad f^{12}(X,Y) \mapsto a(X)a(Y) f^{12}(X,Y),\\
&f^{21}(X,Y) \mapsto \frac{1}{a(X)a(Y)} f^{21}(X,Y)
\end{aligned}
\tag{4.12}
$$

leaves qdet unchanged. Choosing $a(X) = e^{X^{1+\mu}}$ $(0 < \mu \ll 1)$ we see from (4.10) that each term is bounded for $X, Y \to \infty$. According to Hadamard's bound for determinants, the dependence on k is therefore bounded by $k^{k/2}M^k$ for some $M > 0$. $\square$

Because each I_r is bounded from below, the bound (4.9) establishes (4.6). The remaining task is to verify (4.7). First we make note of an alternative form of $f^{22}(X,Y)$.

Lemma 11. *The formula for $f^{22}(X,Y)$ in (1.14) can be written*

$$
\begin{aligned}
f^{22}(X,Y) \;=\; & K^{\mathrm{soft}}(X,Y) + \frac{1}{2}\Big(\mathrm{Ai}(Y) + \frac{u}{2}\int_Y^\infty \mathrm{Ai}(s)\,ds \Big)\\
& \times e^{uX/2}\Big(e^{-u^3/24} - \int_X^\infty e^{-ut/2}\mathrm{Ai}(t)\,dt \Big).
\end{aligned}
\tag{4.13}
$$

Proof. This follows by making use of the integral formula in (1.3) for K^{soft}, integration by parts, and use of the Fourier transform

$$
\int_{-\infty}^{\infty} e^{-yx}\mathrm{Ai}(x)\,dx = e^{-y^3/3}. \tag{4.14}
$$

$\square$

It is precisely the form (4.13) which appears in the asymptotic form of $\bar{S}(x,y)$ relevant to verifying (4.7).

Proposition 12. *We have*

$$
Q^{1/6}\bar{S}(2\sqrt{Q} + Q^{1/6}X, 2\sqrt{Q} + Q^{1/6}Y)\Big|_{\sqrt{\alpha}=1-2w/Q^{1/6}} = f^{22}(X,Y)\Big|_{u=-4w} + O(Q^{-1/6})O(e^{-Y}). \tag{4.15}
$$

Proof. Our main tool is the $\nu \to \infty$ asymptotic expansion

$$J_\nu(\nu - x(\nu/2)^{1/3}) \sim \Big(\frac{2}{\nu}\Big)^{1/3} \mathrm{Ai}(x) + O(\frac{1}{\nu})O(e^{-x}), \tag{4.16}$$

uniform in $x > x_0$ (x_0 fixed). As noted in [6, Eq. (4.11)], this is a consequence of results due to Olver [25]. Using this with

$$\nu = 2\sqrt{Q} + Q^{1/6}X + n, \qquad x = X + nQ^{-1/6}$$

shows

$$\begin{aligned}
&\sum_{n=1}^\infty J_{n+x}(2\sqrt{Q})J_{n+y}(2\sqrt{Q})\Big|_{\substack{x=2\sqrt{Q}+Q^{1/6}X\\ y=2\sqrt{Q}+Q^{1/6}Y}} \\
&\quad = \frac{1}{Q^{1/3}}\sum_{n=1}^\infty \Big(\mathrm{Ai}(X+nQ^{-1/6}) + O(Q^{-1/6})O(e^{-(X+nQ^{-1/6})})\Big) \\
&\quad \times \Big(\mathrm{Ai}(Y+nQ^{-1/6}) + O(Q^{-1/6})O(e^{-(Y+nQ^{-1/6})})\Big) \\
&\quad = \frac{1}{Q^{1/6}}\Big(\int_0^\infty \mathrm{Ai}(X+t)\mathrm{Ai}(Y+t)\,dt + O(Q^{-1/6})O(e^{-(X+Y)})\Big),
\end{aligned} \tag{4.17}$$

where to obtain the final equality use has been made of the fact that a Riemann sum appears in the previous equality.

The sum over j in the second term of (3.35) is not suited to the use of (4.16). To overcome this, we apply the generating function expansion

$$\sum_{n=-\infty}^\infty t^n J_n(z) = \exp\Big(\frac{z}{2}(t - \frac{1}{t})\Big)$$

to conclude

$$\sum_{j=0}^\infty \alpha^{j/2} J_{-j+x}(2\sqrt{Q}) = \alpha^{x/2}\Big(e^{-\sqrt{Q}(\sqrt{\alpha}-1/\sqrt{\alpha})} - \sum_{j=1}^\infty \alpha^{-(j+x)/2} J_{j+x}(2\sqrt{Q})\Big).$$

We can now use (4.16) to deduce the asymptotic formula

$$\begin{aligned}
&\sum_{j=0}^\infty \alpha^{j/2} J_{-j+x}(2\sqrt{Q})\Big|_{\substack{x=2\sqrt{Q}+Q^{1/6}X\\ \sqrt{\alpha}=1-2w/Q^{1/6}}} \\
&\quad = e^{8w^3/3}e^{-2wX}O(e^{-2Xw^2/Q^{1/6}}) - \int_0^\infty e^{2wt}\mathrm{Ai}(X+t)\,dt + O(Q^{-1/6})O(e^{-X}).
\end{aligned} \tag{4.18}$$

It remains to consider the sum over l in the final line of (3.35). For this we require in addition to (4.16) the uniform asymptotic expansion [6, Below (4.12)] (again a consequence of results in [25])

$$J_{\nu\pm1}(\nu - x(\nu/2)^{1/3}) \sim \Big(\frac{2}{\nu}\Big)^{1/3}\mathrm{Ai}(x) \pm \Big(\frac{2}{\nu}\Big)^{2/3}\mathrm{Ai}'(x) + O(\frac{1}{\nu})O(e^{-x}). \tag{4.19}$$

We find

$$\begin{aligned}
&\sum_{l=0}^\infty \Big(J_{2l+2+y}(2\sqrt{Q}) - \sqrt{\alpha}J_{2l+1+y}(2\sqrt{Q})\Big)\Big|_{\substack{x=2\sqrt{Q}+Q^{1/6}X\\ y=2\sqrt{Q}+Q^{1/6}Y\\ \sqrt{\alpha}=1-2w/Q^{1/6}}} \\
&\quad = \frac{1}{2Q^{1/6}}\Big(2w\int_0^\infty \mathrm{Ai}(Y+t)\,dt - \mathrm{Ai}(Y) + O(Q^{-1/3})O(e^{-Y})\Big).
\end{aligned} \tag{4.20}$$

Substituting (4.17), (4.18) and (4.20) in (3.35) gives the stated result. □

For the remaining matrix elements in (1.14) the following forms appear in the asymptotic form of $\bar{D}$ and $\bar{I}$.

Lemma 13. *The formulas for f^{12} and f^{21} in (1.14) can be written*

$$\begin{aligned}
f^{12}(X,Y) &= \frac{1}{4}\Big(\frac{u}{2}+\frac{\partial}{\partial X}\Big)\Big(\frac{u}{2}+\frac{\partial}{\partial Y}\Big) \\
&\quad \times \int_0^\infty ds \int_s^\infty dt\, \Big(\mathrm{Ai}(Y+s)\mathrm{Ai}(X+t) - \mathrm{Ai}(X+s)\mathrm{Ai}(Y+t)\Big), \\
f^{21}(X,Y) &= -e^{u|X-Y|/2}\mathrm{sgn}(X-Y) \\
&\quad +\Big(\int_Y^\infty e^{u(Y-t)/2}K^{\mathrm{soft}}(X,t)\,dt - \int_X^\infty e^{u(X-t)/2}K^{\mathrm{soft}}(Y,t)\,dt\Big) \\
&\quad -e^{u(Y-X)/2}e^{-u^3/24}\int_X^\infty e^{us/2}\mathrm{Ai}(s)\,ds + e^{u(X-Y)/2}e^{-u^3/24}\int_Y^\infty e^{us/2}\mathrm{Ai}(s)\,ds.
\end{aligned} \tag{4.21}$$

Proof. These follow straightforwardly upon using the integral formula in (1.3) and the Fourier transform (4.14). □

Proposition 14. *Let $0<\mu<1/2$ be fixed. We have*

$$Q^{1/3}\bar{D}(2\sqrt{Q}+Q^{1/6}X, 2\sqrt{Q}+Q^{1/6}Y)\Big|_{\sqrt{\alpha}=1-2w/Q^{1/6}} = f^{12}(X,Y)\Big|_{u=-4w} + O\Big(\frac{1}{Q^{1/6}}\Big)O(e^{-X^{1+\mu}-Y^{1+\mu}}),$$

$$\begin{aligned}
&\bar{I}(2\sqrt{Q}+Q^{1/6}X, 2\sqrt{Q}+Q^{1/6}Y) \\
&\quad = f^{21}(X,Y)\Big|_{u=-4w} + e^{-2w|Y-X|}O(e^{2w^2|Y-X|/Q^{1/6}}-1) + O\Big(\frac{1}{Q^{1/6}}\Big)\Big(O(e^{-X})+O(e^{-Y})\Big).
\end{aligned}$$

Proof. The procedure used in the proof of Proposition 12 suffices, with the error terms in (4.16) and (4.19) sharpened from $O(e^{-X})$ to $O(e^{-X^{1+\mu}})$. □

We remarked below the definition (4.11) of a quaternion determinant that only specific combinations of the elements in the underlying 2×2 matrix can occur. We can use this fact together with the asymptotic expansions from Propositions 12 and 14 to deduce an asymptotic formula relating $\rho_{(k)}^{(Q,\alpha)}$ to $\rho_{(k)}^{\mathrm{scaled}}$. This in turn implies the $Q\to\infty$ limit of (4.4).

Corollary 3. *For $Q\to\infty$ we have*

$$\begin{aligned}
&Q^{k/6}\rho_{(k)}^{(Q,\alpha)}(2\sqrt{Q}+Q^{1/6}X_1,\dots,2\sqrt{Q}+Q^{1/6}X_k)\Big|_{\sqrt{\alpha}=1-2w/Q^{1/6}} \\
&\quad = \rho_{(k)}^{\mathrm{scaled}}(X_1,\dots,X_k;-4w) + O\Big(\frac{1}{Q^{1/6}}\Big)O(e^{-(X_1+\cdots+X_k)}).
\end{aligned} \tag{4.22}$$

Consequently, with I_r specified as in (4.5) and $s_0 := \infty$,

$$\begin{aligned}
&\lim_{Q\to\infty}\Big\langle \prod_{l=1}^\infty\Big(1-\sum_{r=1}^k \xi_r\chi_{\lambda_l-l\in I_r}\Big)\Big\rangle^{(\alpha,Q)}\Big|_{\sqrt{\alpha}=1-2w/Q^{1/6}} = 1 \\
&+\sum_{p=1}^\infty \frac{(-1)^p}{p!}\int_{-\infty}^\infty dX_1\dots\int_{-\infty}^\infty dX_p \prod_{i=1}^p\Big(\sum_{r=1}^k \xi_r\chi^{(i)}_{(s_r,s_{r-1})}\Big)\rho_{(p)}^{\mathrm{scaled}}(X_1,\dots,X_p;-4w).
\end{aligned} \tag{4.23}$$

Proof. Consider first (4.22). The leading term is immediate. Regarding the error bound, as implied by the remark below (4.11), the elements in the 2×2 matrix of (3.31) appear as factors in (4.11) only in the combinations $\bar{S}(x,x)$, $\bar{S}(x,y)\bar{S}(y,x')$ or $\bar{I}(x,y)\bar{D}(y,x')$. This, together with the individual error bounds in Propositions 12 and 14, imply the error bound in (4.22).

We know from Proposition 9 that to establish (4.23) it suffices to establish (4.6) and (4.7). As previously remarked, the bound (4.9) implies (4.6). Thus it remains to check that with a_j as in (4.5),

$$\lim_{Q\to\infty} Q^{k/6} \sum_{h_1,\dots,h_p=0}^{\infty} \prod_{i=1}^{p} \chi^{(i)}_{(a_r,a_{r-1})} \rho^{(Q,\alpha)}_{(p)}(h_1,\dots,h_p)\Big|_{\sqrt{\alpha}=1-2w/Q^{1/6}}$$
$$= \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_p \prod_{i=1}^{p} \chi^{(i)}_{s_r,s_{r-1}} \rho^{\rm scaled}_{(p)}(x_1,\dots,x_p;-4w). \tag{4.24}$$

For this we note

$$\sum_{h_1,\dots,h_p=0}^{\infty} \prod_{i=1}^{p} \chi^{(i)}_{(2\sqrt{Q}+Q^{1/6}s_r,2\sqrt{Q}+Q^{1/6}s_{r-1})} \rho^{(Q,\alpha)}_{(p)}(h_1,\dots,h_p)$$
$$= \sum_{\substack{x_1,\dots,x_p:\\ 2\sqrt{Q}+Q^{1/6}x_j\in\mathbb{Z}}} \prod_{i=1}^{p} \chi^{(i)}_{(s_r,s_{r-1})} \rho^{(Q,\alpha)}_{(p)}(2\sqrt{Q}+Q^{1/6}x_1,\dots,2\sqrt{Q}+Q^{1/6}x_p).$$

Recognising this as a Riemann sum, and substituting (4.22) gives (4.24). $\square$.

4.3 Proof of theorem 1

According to (4.2), (4.3), (2.7) and (2.8)

$$\lim_{N\to\infty} \Pr(\lambda^{(1)}_{n,m}\le a_1,\dots,\lambda^{(l)}_{n,m}\le a_l)\Big|_{\substack{a_j=2\sqrt{Q}+Q^{1/6}s_j\\ n=[Q/2]\\ m=[\sqrt{Q}-2wQ^{1/3}]}}$$
$$= \lim_{Q\to\infty} \Pr{}^{(Q,\alpha)}(\lambda_1\le a_1,\dots,\lambda_l\le a_l)\Big|_{\substack{a_j=2\sqrt{Q}+Q^{1/6}s_j\\ \sqrt{\alpha}=1-2w/Q^{1/6}}}$$
$$= \sum_{(n_1,\dots,n_l)\in\mathbb{L}_l} \frac{(-1)^{\sum_{r=1}^{l}n_r}}{n_1!\cdots n_l!} \frac{\partial^{\sum_{j=1}^{l}n_j}}{\partial\xi^{n_1}\cdots\partial\xi^{n_l}} \lim_{Q\to\infty} \Big\langle \prod_{j=1}^{\infty}\Big(1-\sum_{r=1}^{k}\xi_r\chi_{\lambda_j-j\in I_r}\Big)\Big\rangle^{(\alpha,Q)}\Big|_{\alpha=1-2w/Q^{1/6}}. \tag{4.25}$$

The limit in the final expression is evaluated according to (4.23), which shows (4.25) can be written

$$\sum_{(n_1,\dots,n_l)\in\mathbb{L}_l} E^{(w)}(\{n_r,(s_r,s_{r-1})\}_{r=1,\dots,l}), \tag{4.26}$$

where $E^{(w)}(\{n_r,(s_r,s_{r-1})\}_{r=1,\dots,l})$ denotes the probability that the interval (s_r,s_{r-1}) contains no particles in the point process specified by the correlations (1.14) with w given by (1.11). But (4.26) is, according to (2.7), equal to the distribution function $F^{\boxslash}(s_1,\dots,s_l;w)$ appearing on the right hand side of (1.13).

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